

DECOHERENCE OF ANOMALOUSLY-FLUCTUATING STATES OF FINITE MACROSCOPIC SYSTEMS

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In quantum systems of a macroscopic size V , such as interacting many particles and quantum computers with many qubits, there exist pure states such that fluctuations of some intensive operator \hat{A} is anomalously large, $\langle \delta \hat{A}^2 \rangle = \mathcal{O}(V^0)$, which is much larger than that assumed in thermodynamics, $\langle \delta A^2 \rangle = \mathcal{O}(1/V)$. By making full use of the locality, we show, starting from Hamiltonians of macroscopic degrees of freedom, that such states decohere at anomalously fast rates when they are weakly perturbed from environments.

I. INTRODUCTION

We consider a quantum system, which extends spatially over a *macroscopic but finite* volume V . Such a system includes, for example, many particles confined in a box of volume V , and a system composed of N two-level systems (qubits), for which $V \propto N \gg 1$. Such a system, in general, has pure states such that fluctuations of some intensive operator,

$$\hat{A} = \frac{1}{V} \sum_{x \in V} \hat{a}(x), \quad (1)$$

where $\hat{a}(x)$ is an operator at point x , is anomalously large;

$$\langle \delta \hat{A}^2 \rangle = \mathcal{O}(V^0). \quad (2)$$

We call such a pure state an ‘anomalously fluctuating state’ (AFS), because $\langle \delta \hat{A}^2 \rangle$ is much larger than that assumed in thermodynamics, $\langle \delta A^2 \rangle = \mathcal{O}(1/V)$.

In *closed* quantum systems, AFSs appear naturally, as explained in section II. Experimentally, however, it is rare to encounter AFSs. This apparent contradiction may be explained by the fact that real physical systems are not completely closed: Interactions with environments would destroy AFSs very quickly. Effects of environments have been discussed intensively in studies of ‘macroscopic quantum coherence’ [1], and of quantum measurement [2]. These previous studies assumed that the principal systems of interest were describable by a *small number* of collective coordinates, which interact *non-locally* with

some environment(s). However, justification of these assumptions is not clear. Although such models might be applicable to systems which have a non-negligible energy gap to excite ‘internal coordinates’ in the collective coordinates, there are many systems which do not have such an energy gap. Moreover, the results depended strongly on the choices of the coordinates and the form of the nonlocal interactions, so that general conclusions were hard to draw. In this work, we study the decoherence rates of AFSs and normally-fluctuating states (NFSs), starting from microscopic Hamiltonians of macroscopic degrees of freedom, by making full use of the locality: interactions must be local (Eq. (3)), and macroscopic variables must be averages over a macroscopic region (Eq. (1)). To express the locality manifestly, we use a local field theory throughout this work.

II. ANOMALOUSLY-FLUCTUATING STATES

AFSs generally appear in, e.g., (i) finite systems which will exhibit symmetry breaking if V goes to infinity, and (ii) quantum computers with many qubits.

In case (i), we can find states (of finite systems) which approach a symmetry-breaking vacuum as $V \rightarrow \infty$. We call such a state a pure-phase vacuum (PPV). It has a finite expectation value $\langle \hat{M} \rangle = \mathcal{O}(V^0)$ of an order parameter \hat{M} , and has negligible fluctuations $\langle \delta \hat{A}^2 \rangle = \mathcal{O}(1/V)$ for *any* intensive operator \hat{A} (including \hat{M}) [3–8].

Hence, PPVs are NFSs. In a mean-field approximation, PPVs have the lowest energy. However, it is known that the exact lowest-energy state of a finite system (without a symmetry-breaking field) is generally the *symmetric* ground state (SGS), for which $\langle \hat{M} \rangle = 0$ [3–8]. PPVs have a higher (or, in some special cases, equal) energy than the SGS. The SGS is composed primarily of a superposition of PPVs with different values of $\langle \hat{M} \rangle$ [3–8]. As a result, it has an anomalously large fluctuation of \hat{M} ; $\langle \delta \hat{M}^2 \rangle = \mathcal{O}(V^0)$. Therefore, if one obtains the exact lowest-energy state (e.g., by numerical diagonalization) in case (i), it is generally an AFS.

In case (ii), various states appear in the course of a quantum computation. Some state may be an NFS, for which $\langle \delta \hat{A}^2 \rangle = \mathcal{O}(1/V)$ for *any* intensive operator A . This means that correlations between distant qubits are weak [8,9]. Properties of such states may be possible to emulate by a classical system with local interactions, because entanglement is weak. We therefore conjecture that other states – AFSs – should appear in some stage of the computation for a quantum computer to be much faster than classical computers. In fact, we confirmed this conjecture in Shor’s algorithm for factoring [9].

We stress that AFSs are peculiar to *quantum* systems of macroscopic but *finite* sizes, and thus are very interesting. In (local) classical theories, a state such that $\langle \delta \hat{A}^2 \rangle = \mathcal{O}(V^0)$ is possible only as a mixed state. In (local) quantum theory of infinite systems, any pure states (including excited states) are NFSs [3,4], because all AFSs for a finite V become mixed states in the limit of $V \rightarrow \infty$ [3,4,7].

III. INTERACTING MANY BOSONS

We first consider interacting many bosons confined in a uniform box of volume V with the periodic boundary conditions [6,7,10]. Since the Hamiltonian \hat{H} commutes with the number of bosons \hat{N} , there exist simultaneous eigenstates of \hat{N}

and \hat{H} . We denote the lowest-energy state for a given value of N as $|N, G\rangle$. This is the SGS, for which $\langle \hat{\Psi} \rangle = 0$, where $\hat{\Psi}$ denotes the intensive order parameter $\hat{\Psi} \equiv (1/V) \sum_{x \in V} \hat{\psi}(x)$, and $\hat{\psi}(x)$ is the boson operator at point x in the box. On the other hand, $|N, G\rangle$ has the long-range order, $\langle \hat{\psi}^\dagger(x) \hat{\psi}(x') \rangle = \mathcal{O}(V^0)$ for $|x - x'| \sim V^{1/3}$ [7,10]. We can easily show that it has an anomalously-large fluctuation of $\hat{\Psi}$; $\langle \delta \hat{\Psi}^\dagger \delta \hat{\Psi} \rangle = \mathcal{O}(V^0)$, which shows that $|N, G\rangle$ is an AFS. On the other hand, by superposing $|N, G\rangle$ of various values of N [11], we can construct a state $|\alpha, G\rangle$, for which $\langle \hat{\Psi} \rangle = \mathcal{O}(V^0)$ and $\langle \delta \hat{A}^2 \rangle = \mathcal{O}(1/V)$ for *any* intensive operator A [7]. Namely, $|\alpha, G\rangle$ is a PPV, hence is an NFS. Although it is not an eigenstate of \hat{H} , $|\alpha, G\rangle$ does not collapse for a macroscopic time [6].

Since the energy of $|N, G\rangle$ is lower (by $\mathcal{O}(V^0)$) than that of $|\alpha, G\rangle$ for the same value of $\langle \hat{N} \rangle$ [6], there seems to be nothing against the realization of $|N, G\rangle$ if the system is closed. However, most real systems are not completely closed, and weak perturbations from environments can alter the situation dramatically. In fact, effects of interactions with an environment, which was assumed to be a huge room that has initially no bosons, on these states was studied in Ref. [10], where it was shown that $|N, G\rangle$ decoheres much faster than $|\alpha, G\rangle$, as bosons escape from the box into the environment. Namely, the SGS (which is an AFS) is much more fragile than the PPV (an NFS), for interacting many bosons in a leaky box. This may be the first example in which the SGS and PPVs are identified *and* the fragility of the SGS as well as the robustness of PPVs are shown, for a non-trivial interacting many-particle system.

IV. GENERAL SYSTEMS

We next consider a *general* finite system of a *large* V , interacting with a *general* environment E , with a *local* interaction,

$$\hat{H}_{\text{int}} = \lambda \sum_{x \in V_G} \hat{a}(x) \otimes \hat{b}(x), \quad (3)$$

where $\hat{a}(x)$ and $\hat{b}(x)$ are local operators (*any* functions of the fields and their conjugate momenta at point x) of the principal system and E, respectively, *at the same point* $x \in V_C$. Here, $V_C (\subseteq V)$ is a ‘contact region’ between the principal system and E. Since we are interested in the case of weak perturbations from E, we assume that the coupling constant $\lambda (\geq 0)$ is small.

The initial ($t = 0$) state, $\rho(0) = |\phi\rangle\langle\phi|$, of the principal system is either an AFS or an NFS, and the initial state of the total system is assumed to be the uncorrelated product, $\rho_{\text{tot}}(0) = \rho(0) \otimes \rho_E$, where ρ_E is a time-invariant state of E. We are interested in the time evolution of the reduced density operator, $\rho(t) \equiv \text{Tr}_E[\rho_{\text{tot}}(t)]$, which generally evolves from the pure state $|\phi\rangle\langle\phi|$ into a mixed state. As a measure of the purity, we evaluate the ‘linear entropy’ defined by $S_{\text{lin}}(t) \equiv 1 - \text{Tr}[\rho(t)^2]$, as a power series of λ^2 , $S_{\text{lin}} = S_{\text{lin}}^{(0)} + S_{\text{lin}}^{(1)} + \dots$, where $S_{\text{lin}}^{(n)} = \mathcal{O}(\lambda^{2n})$. We confirmed that this series converges [8]. Since $S_{\text{lin}}^{(0)} = 0$, this suggests that $S_{\text{lin}}^{(1)}$ would give the dominant contribution to S_{lin} under our assumption that λ is small.

If $|\phi\rangle$ is translationally invariant [12], both spatially and temporally, we can show that [8]

$$S_{\text{lin}}^{(1)}(\phi, t) \geq \frac{\lambda^2}{\hbar^2} g_{00} \langle \phi | \delta \hat{A}^\dagger \delta \hat{A} | \phi \rangle t, \quad (4)$$

where $\hat{A} \equiv (1/V) \sum_{x \in V} \hat{a}(x)$, and g is a positive matrix defined by the time correlation of \hat{b} of E;

$$g_{k_1 k_2} \equiv \frac{1}{2} \int_{-\infty}^{\infty} ds \langle \hat{b}_{k_1}^\dagger \hat{b}_{k_2}(s) \rangle. \quad (5)$$

Here, $b_k \equiv \sum_{x \in V_C} b(x) e^{-ikx}$, where the sum is not over the entire region of E, but over V_C . Since the rhs of Eq. (4) is proportional to t , we can interpret it divided by t as a *lower bound of the decoherence rate*, which we denote γ . It is proportional to the fluctuation of the intensive operator \hat{A} composed of $\hat{a}(x)$ which constitutes H_{int} . As discussed in the next section, in

real physical systems many terms would exist in \hat{H}_{int} ; $\hat{H}_{\text{int}} = \hat{H}_{\text{int}}^{[1]} + \hat{H}_{\text{int}}^{[2]} + \dots$, where $\hat{H}_{\text{int}}^{[\ell]} = \lambda^{[\ell]} \sum_{x \in V_C^{[\ell]}} \hat{a}^{[\ell]}(x) \otimes \hat{b}^{[\ell]}(x)$. It may be possible to construct a state which is *exactly robust* (i.e., does not decohere at all) against *one* of $\hat{H}_{\text{int}}^{[\ell]}$ ’s. Such a state, however, would be fragile against another $\hat{H}_{\text{int}}^{[\ell]}$. It is therefore important to judge the fragility and robustness against *all* local interactions. To γ of an NFS, all of them give only small contributions, except when $g_{00} \geq \mathcal{O}(V)$, because $\hat{H}_{\text{int}}^{[\ell]}$ gives

$$\gamma^{[\ell]} = (\lambda^{[\ell]2}/\hbar^2) g_{00}^{[\ell]} \times \mathcal{O}(1/V), \quad (6)$$

which is small for a large V . To γ of an AFS, on the other hand, some of $\hat{H}_{\text{int}}^{[\ell]}$ ’s can give an anomalously large contribution, because, by definition, $\langle \phi | \delta \hat{A}^{[\ell]\dagger} \delta \hat{A}^{[\ell]} | \phi \rangle = \mathcal{O}(V^0)$ for some $\hat{A}^{[\ell]}$, hence

$$\gamma^{[\ell]} = (\lambda^{[\ell]2}/\hbar^2) g_{00}^{[\ell]} \times \mathcal{O}(V^0), \quad (7)$$

which is larger than Eq. (6) by a macroscopic factor $\mathcal{O}(V)$. To see more details, we estimate the prefactor g_{00} [8]. Let V_E^{corr} be the size of the region in which $\int_{-\infty}^{\infty} dt \langle b^*(x) b(0, t) \rangle$ is correlated in E. We can roughly estimate that $g_{00} \propto V_C^2$ when $V_E^{\text{corr}} > V_C$, whereas $g_{00} \propto V_C V_E^{\text{corr}}$ when $V_E^{\text{corr}} < V_C$ [8]. In either case, the rhs of Eq. (7) becomes macroscopically large if the contact region V_C is macroscopically large. Namely, AFSs are fragile. On the other hand, unlike the case of section III, we cannot draw a general conclusion on the robustness of NFSs for general systems.

V. EFFECTIVE THEORIES

Usually, we are only interested in phenomena in some energy range ΔE . Hence, it is customary to analyze a physical system by an effective theory which correctly describes the system only in ΔE . In some cases, the degrees of freedom N of the effective theory can become small even for a macroscopic system when, e.g., a non-negligible energy gap exists in ΔE because

then the number of quantum states in ΔE can be small. Some SQUID systems are such examples. We here exclude such systems, and concentrate on systems whose N is a macroscopic number, because otherwise the difference between $\mathcal{O}(1/V)$ and $\mathcal{O}(V^0)$ would be irrelevant (where $V \propto N$). The effective theory can be constructed from an elementary dynamics by an appropriate renormalization process. In this process, in general, many interaction terms would be generated in the effective interaction H_{int} [13]. Hence, it seems rare that H_{int} does not have any term which takes the form of Eq. (3) such that $\langle \delta A^\dagger \delta A \rangle = \mathcal{O}(V^0)$ for the AFS under consideration.

If N were small, one could drop terms with small $\lambda^{[\ell]}$. Since N is large, however, such an approximation can be wrong, because, as discussed in the previous section, its contribution may be enhanced by a macroscopic factor and become relevant to AFSs, however small $\lambda^{[\ell]}$ is.

VI. IMPLICATIONS

We finally discuss implications of the above results, for (i) and (ii) of section II.

In case (i), our results suggest a new origin of symmetry breaking in finite systems [10]. Although symmetry breaking is usually described as a property of infinite systems, it is observed in finite systems as well. Our results suggest that although a PPV (which is an NFS) has a *higher* energy than the SGS (an AFS), the former is realized because the latter is extremely fragile in environments. This scenario, may be called ‘environment-induced symmetry breaking’ after Zurek’s ‘environment-induced superselection rule’ [2], may be the physical origin of symmetry breaking in finite systems.

In case (ii), our results show that the decoherence rate can be estimated by fluctuations of intensive operators, which depend strongly on the number of qubits N and the natures of the states of the qubits [14]. This may become a key to the fight against decoherence in quantum computation.

In both cases, we stress that the *approx-*

imate robustness against *all* local interactions (between the principal system and environments) would be more important than the *exact* robustness against a particular interaction, because, as discussed in section IV, many types of interactions would coexist in real physical systems, and the exact robustness against one of them could imply fragility to another.

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